The Hamiltonian is now

$$H = \mu(B - R_v)\left(\frac{1}{\sqrt{B-2R_v}} - \frac{\mu G}{2R}\frac{1}{\sqrt{{R_v}^2 - 2R_v + B}}\right). \tag{6}$$

where $B = 1 - R_s/R$, $R_s = 2GM$, and $R_v = \partial R/\partial v$. We emphasis that so far we did not use any approximations, so the Hamiltonian in Eq. (6) is exact.

III. QUANTUM COLLAPSE OF THE SHELL IN THE LIMIT OF $R \to 0$

The main goal of the this paper is to see what happens at the last stages of the collapse of the shell, i.e. when $R \to 0$. Since we have an explicit Hamiltonian of the system, we can apply the functional Schrodinger formalism and study quantum effects near the classical singularity. In the framework of the functional Schrodinger formalism, we will simply write down the Schrodinger equation for the wave-functional $\Psi[R(v)]$, and try to solve it.

We first derive the behavior of R_τ near $R = 0$. From Eq. (4), we have

$$R_\tau = \sqrt{\left(\frac{M}{\mu} + \frac{\mu G}{2R}\right)^2 - 1}. \tag{7}$$

From here we see that $R_\tau \approx \frac{\mu G}{2R}$ as R is approaching zero. Substituting this result in Eq.(7), we find

$$R_v \approx -\frac{1}{2}\left(\frac{\mu G}{R}\right)^2 \tag{8}$$

Thus, the rate at which the dust shell collapses near $R = 0$ diverges as $R_v \propto \frac{1}{R^2}$.

In this limit the Hamiltonian in Eq. (6) can be approximated as

$$H = \mu(-R_v)\left[\frac{1}{\sqrt{2\mid R_v\mid}} - \frac{\mu G}{2R}\frac{1}{R_v}\right] \tag{9}$$

which gives

$$R_v = 2\left(\frac{H}{\mu} - \frac{G\mu}{2R}\right)^2 \tag{10}$$

For $R \to 0$, we can ignore the constant term H/μ, and we will again get Eq. (8).

In the limit $R \to 0$, the canonical momentum reduces to

$$\Pi = \mu\left[\frac{1}{\sqrt{2\mid R_v\mid}} + \frac{\mu G}{2R}\right] \tag{11}$$

Expressing R_v in terms of Π in Hamiltonian (9) we get

$$H = \frac{-R}{G}\left[1 - \frac{2\Pi R}{\mu^2 G}\right]^{-1} + \frac{\mu^2 G}{2R}. \tag{12}$$

The Hamiltonian in Eq. (12) governs the evolution of the collapsing dust shell in vicinity of $R = 0$. As in the standard quantization procedure, we promote the momentum Π into an operator

$$\Pi = -i\hbar\frac{\partial}{\partial R}. \tag{13}$$

We can now write the functional Schrodinger equation for the wave-functional $\psi[R(v)]$

$$H\psi = i\hbar\frac{\partial \psi}{\partial v} \tag{14}$$

and try to solve it. Unfortunately, the structure of the Hamiltonian (12) is such that the usual treatment is practically impossible. The main problem is that the differential operator in Hamiltonian (12) is non-local. This finding represents a strong support for suggestions that quantum gravity might be ultimately a non-local theory.

While finding solutions to non-local equations is very difficult, we will show that it is possible to define a procedure (similar to the one outlined in [5]) which will lead to the solution of Eq. (14). We first isolate the non-local operator $\hat{A}$ from the Hamiltonian (12)

$$\hat{A} = \left[1 - \frac{2\Pi R}{\mu^2 G}\right]^{-1} \tag{15}$$

Its inverse is

$$\hat{A}^{-1} = 1 - \frac{2\Pi R}{\mu^2 G} \tag{16}$$

We can take care of the operator ordering as

$$\hat{A} = \left[1 - \frac{1}{\mu^2 G}(\hat{\Pi}R + R\hat{\Pi})\right]^{-1}, \tag{17}$$

so that

$$\hat{A}^{-1} = 1 - \frac{1}{\mu^2 G}(\hat{\Pi}R + R\hat{\Pi}). \tag{18}$$

In terms of derivatives, $\hat{A}^{-1}$ is

$$\hat{A}^{-1} = (1 + \frac{i}{\mu^2 G}) + \frac{2iR}{\mu^2 G}\frac{\partial}{\partial R} \tag{19}$$

Let's define the action of an operator $\hat{A}$ as $\varphi = \hat{A}\psi$, which means $\psi = \hat{A}^{-1}\varphi$, where φ is just some function which gives the wavefunction ψ upon action of the operator $\hat{A}$. Explicit action of $\hat{A}^{-1}$ on φ converts the equation $\hat{A}^{-1}\varphi = \psi$ into a linear differential equation

$$\frac{d\varphi}{dR} + \frac{1}{2R}(1 - i\mu^2 G)\varphi + \frac{i\mu^2 G}{2R} = 0 \tag{20}$$

This equation can be solved to give

$$\varphi = -\frac{i\mu^2 G}{2}\frac{\int R^{-\frac{(1+i\mu^2 G)}{2}}\psi dR}{R^{\frac{(1-i\mu^2 G)}{2}}} \tag{21}$$

Since $\varphi = \hat{A}\psi$ we obtain the action of $\hat{A}$ as

$$\hat{A} = -\frac{i\mu^2 G}{2}\frac{\int R^{-\frac{(1+i\mu^2 G)}{2}}(.)dR}{R^{\frac{(1-i\mu^2 G)}{2}}} \tag{22}$$

where (.) is the placeholder for the function on which $\hat{A}$ is acting.

Let's concentrate on the stationary solutions to Eq. (14) in the form of

$$\psi(R,v) = \psi(R)e^{iEv/\hbar} \tag{23}$$

where v is the time evolution parameter, and E is the energy eignevalue. The time independent Schrodinger equation becomes $H\psi = E\psi$. The Hamiltonian in Eq. (12) in terms of the operator $\hat{A}$ becomes

$$H = \frac{R\hat{A}}{G} + \frac{\mu^2 G}{2R} \tag{24}$$

Accounting for the ordering of operators, this Hamiltonian becomes

$$H = -\frac{1}{2G}\left(R\hat{A} + \hat{A}R\right) + \frac{\mu^2 G}{2R}. \tag{25}$$

The Schrodinger equation (14) becomes

$$\frac{-1}{2G}\left(R\hat{A} + \hat{A}R\right) + \frac{\mu^2 G}{2R} = E\psi \tag{26}$$

When $\hat{A}$ operates on R we get

$$\hat{A}(R) = -\frac{\alpha}{2}R^{-\frac{1}{2}(1-\alpha)}\int R^{-\frac{1}{2}(1+\alpha)}RdR \tag{27}$$

which yields

$$\hat{A}(R) = \frac{-\alpha R + \beta}{3-\alpha} \tag{28}$$

where $\alpha = i\mu^2 G$ and β is an integration constant. So our equation becomes

$$\frac{-1}{2G}\left(R\hat{A}\psi + \frac{\alpha R - \beta}{\alpha - 3}\right) + \frac{\mu^2 G}{2R} = E\psi \tag{29}$$

Now we can move all the terms to one side and separate the term with the integral

$$\int R^{-\frac{1}{2}(1+\alpha)}\psi dR = \frac{4GR^{-\frac{1+\alpha}{2}}}{\alpha}\left[\frac{1}{2G}\left(\frac{\alpha R - \beta}{\alpha - 3}\right) - \frac{\mu^2 G}{2R} + E\right] \tag{30}$$

We can now differentiate this equation with respect to R to remove integration. Differentiation yields

$$\left[\frac{2R^{\frac{1}{2}(1-\alpha)}}{\alpha-3} - 2\mu^2 G^2 R^{-\frac{1}{2}(3+\alpha)} + \left(\frac{4GE}{\alpha} - \frac{2\beta}{\alpha(\alpha-3)}\right)R^{-\frac{1}{2}(1+\alpha)}\right]' = \left[\left(\frac{\alpha-1}{\alpha-3}+1\right)R^{-\frac{1}{2}(1+\alpha)} - \mu^2 G^2(\alpha+3)R^{-\frac{1}{2}(5+\alpha)} + \left(\frac{2GE(\alpha+1)}{\alpha} - \frac{\beta(\alpha+1)}{\alpha(\alpha-3)}\right)R^{-\frac{1}{2}(3+\alpha)}\right] \tag{31}$$

This can be written as

$$\frac{d\psi}{} = \int \frac{a_1 + a_2 R + a_3 R^2}{a_4 R + a_5 R^2 + a_6 R^3}dR \tag{32}$$

where $a_1 = -\mu^2 G^2(\alpha+3)$, $a_2 = \left(\frac{2GE(\alpha+1)}{\alpha} - \frac{\beta(\alpha+1)}{\alpha(\alpha-3)}\right)$, $a_3 = 1 + \frac{\alpha-1}{\alpha+3}$, $a_4 = -2\mu^2 G^2$, $a_5 = \left(\frac{4GE}{\alpha} - \frac{2\beta}{\alpha(\alpha-3)}\right)$ and $a_6 = \frac{2}{\alpha-3}$ This integral can be solved for general values of constants. However, since we are working in the limit of $R \approx 0$, we keep only the leading order terms

$$\ln\psi = \int \frac{a_1}{a_4 R}dR + \text{constant} \tag{33}$$

Solving this equation and substituting the values of the constants, we find the solution for the wavefunction

$$= \lambda R^{\frac{3+i\mu^2 G}{2}}. \tag{34}$$

where λ is a constant. The corresponding probability density $P = \psi^*\psi$ is

$$|\psi|^2 = \lambda^2 R^3. \tag{35}$$

This result is very important. It demonstrates that the probability density associated with the wavefunction ψ which describes the collapse of the shell of matter is non-singular near the classical singularity. In fact, the probability density in Eq. (35) vanishes exactly at $R = 0$. It is remarkable that a simple quantum treatment of the gravitational collapse indicates that classical singularity at the center can be removed.

IV. CONCLUSIONS

In this paper we studied quantum aspects of the gravitational collapse near the classical singularity as seen by an infalling observer. Since gravity is the by far the weakest force in nature, we expect that quantum mechanics will significantly modify classical behavior of gravity only in the strong field regimes, e.g. near classical singularities. In the absence of a fully fledged theory of quantum gravity, we worked in the context of the functional Schrodinger formalism applied to a simple gravitational system - collapsing shell of matter. We used the Eddington-Finkelstein space-time foliation which is convenient for studying the question of the black hole formation till the very end where the collapsing shell crosses its own Schwarzschild radius and starts approaching the classical singularity at the center. We derived the conserved quantity with the clear interpretation as the Hamiltonian of the system and quantized the theory. In the $R \to 0$ limit, we found that the equation which describes the quantum evolution of the collapsing shell is strongly non-local. Non-local terms which are usually suppressed in large distance limit, become dominant in the near singularity limit. This conforms some earlier speculations and related studies. As an important step forward, we managed to solve this non-local equation explicitly and found the form of the wavefuction. Remarkably, the wavefunction and its corresponding probability density are non-singular at $R \to 0$. This is an indication that quantization can remove classical singularities from gravity, just as it was the case with the singular electromagnetic Coulomb potential.

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